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Spanwise Distribution of Induced Drag in Subsonic Flow by the Vortex Lattice Method

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Nomenclature

A = aspect ratio
C_{Di} = total induced drag coefficient of surface
C_{DiV} = total induced drag coefficient given by Vortex Lattice Method
C_{Diw} = total induced drag given by wake integral
C_L = total lift coefficient of surface
c = chord length of strip
$\bar{c}$ = average chord length of surface
c_{di} = induced drag coefficient of strip
c_l = lift coefficient of strip
D_d = normalwash factor at bound vortex
D_c = normalwash factor at control point
d = induced drag coefficient of box
e = semiwidth of strip
M = Mach number
p = pressure coefficient of box
S = reference area
s = semispan of surface
w_b = dimensionless normalwash at bound vortex
w_c = dimensionless normalwash at control point
x,y,z = Cartesian coordinates
α = angle of attack
α_i = induced incidence at bound vortex
η = dummy spanwise coordinate

Introduction

GARNER[1] has discussed induced drag and its spanwise distribution in incompressible flow. Following Multhopp,[2,3] we may write the induced drag coefficient as

$$C_{D_i} = \frac{1}{2S}\int_{-s}^{s} c_l c \alpha_i dy \qquad (1)$$

where the so-called induced incidence is

$$\alpha_i = \frac{1}{8\pi}\int_{-s}^{s}\frac{1}{y-\eta}\frac{d}{d\eta}(c_l c)d\eta \qquad (2)$$

Garner has shown that for sweptback wings, the quantity $c_l c \alpha_i$ bears no resemblance to the spanwise distribution of induced drag as suggested by Robinson and Laurmann[4] [Eq. (3.3,43)]. This Note reviews the calculation of the spanwise distribution of induced drag by the Vortex Lattice Method (VLM) and presents some comparisons with results of other methods.

The VLM in its modern form avoids the use of pressure loading functions and has been developed independently by several investigators including Rubbert[5] (who also considered the induced drag calculation), Dulmovits,[6] Hedman,[7] and Belotserkovskii.[8] In the application of this method, a division of the surface(s) into small trapezoidal elements (boxes) arranged in strips parallel to the freestream is made so that surface edges, fold lines, and hinge lines lie on box boundaries, as shown in Fig. 1. A horseshoe vortex is placed on each of the boxes such that the bound vortex of the horseshoe system coincides with the quarter-chord line of the box. The surface boundary condition is a prescribed normalwash (e.g., downwash) applied at the control point of each box. The control point is centered spanwise on the three-quarter chord line of the box. This choice of control point location has been shown by James[9] to be optimum for two-dimensional flow and results in a high degree of accuracy for three-dimensional flow. The influences of all the k vortices are summed for each control point j to obtain the total dimensionless normalwash w_c at the control point

$$w_{cj} = \sum_k D_{cjk} p_k \qquad (3)$$

where D_c is the three-quarter chord control point normalwash factor (given, e.g., by Hedman,[7] Appendix 3), and p is the pressure coefficient at the center of the bound vortex. Writing Eq. (3) for all control points j leads to a system of simultaneous equations whose solution for a prescribed distribution of normalwash leads to the desired distribution of pressure coefficients.

We now introduce the normalwash factor D_b that relates the normalwash w_b at the center of the bound vortex to the pressure coefficient of each horseshoe vortex. It is similar to D_c except that the normalwash induced by a bound vortex on itself is zero. Then, the induced drag coefficient d of box j is given by the Kutta-Joukowski Law to be

$$d_j = W_{bj} p_j \qquad (4)$$

where

$$W_{bj} = \sum_k D_{bjk} p_k \qquad (5)$$

The spanwise distribution of induced drag is found by summing the box drag coefficients along each strip, noting that the box drag coefficient is based on the local box area. The total induced drag coefficient is found by summing the drag over all strips on the surface and noting that the strip drag coefficient is based on the strip area.

Applications

The first application is to a rectangular wing with aspect ratio $A = 2.0$ in incompressible flow. A variety of aerodynamic idealizations of the planform into equal size rectangular boxes was investigated. Convergence of the total drag was obtained using 100 boxes from 5 chordwise and 20 spanwise divisions. However, the converged value of the induced drag was too low: $C_{D_i}/C_L^2 = 0.155375$, which is less than $1/\pi A = 0.159155$ by 2.43%.

The tendency of the VLM to predict low values of induced drag has already been observed by Rubbert.[5] This tendency remains to be explained, as do other aspects of the method,[9] but it suggests that a correcting scale factor be introduced into the spanwise distribution of induced drag that is based on the cross-flow energy in the wake (i.e., the total induced drag). The correcting scale factor is defined as the ratio C_{Diw}/C_{DiV} where C_{Diw} is the wake integral given by Eq. (1) and C_{DiV} is the VLM result.

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Because of the applicability of the VLM to unconventional configurations (e.g., the rotor/wings of Ref. 10), a numerical form of Eq. (1) is desirable. Integrating Eq. (2) by parts leads to

$$\alpha_i = -\frac{1}{8\pi}\int_{-s}^{s}\frac{c_l c}{(y-\eta)^2}\,d\eta \tag{6}$$

which for symmetrical loading becomes

$$\alpha_i = -\frac{1}{8\pi}\int_{0}^{s}\left[\frac{c_l c}{(y-\eta)^2}+\frac{c_l c}{(y+\eta)^2}\right]d\eta \tag{7}$$

where the improper integral is evaluated by taking the finite part as discussed by Mangler.[11] Consider the mth strip having semiwidth e_m and with the centerline located at $\eta = y_m$. Then, assuming the spanwise loading to have a parabolic variation§ across the strip

$$\left(\frac{c_l c}{C_L \bar{c}}\right)_m = a_m\eta^2 + b_m\eta + c_m \tag{8}$$

where

$$a_m = \frac{1}{d_{mi}d_{mo}(d_{mi}+d_{mo})}\times \left\{d_{mi}\left(\frac{c_l c}{C_L\bar{c}}\right)_{m+1} - (d_{mi}+d_{mo})\left(\frac{c_l c}{C_L\bar{c}}\right)_m + d_{mo}\left(\frac{c_l c}{C_L\bar{c}}\right)_{m-1}\right\} \tag{9}$$

$$b_m = \frac{1}{d_{mi}d_{mo}(d_{mi}+d_{mo})}\left\{d_{mo}(2\eta_m + d_{mo})\left[\left(\frac{c_l c}{C_L\bar{c}}\right)_m - \left(\frac{c_l c}{C_L\bar{c}}\right)_{m-1}\right] - d_{mi}(2\eta_m - d_{mi})\left[\left(\frac{c_l c}{C_L\bar{c}}\right)_{m+1} - \left(\frac{c_l c}{C_L\bar{c}}\right)_m\right]\right\} \tag{10}$$

$$c_m = \left(\frac{c_l c}{C_L\bar{c}}\right)_m - a_m\eta_m^2 - b_m\eta_m \tag{11}$$

and

$$d_{mi} = e_m + e_{m-1} \tag{12}$$

$$d_{mo} = e_m + e_{m+1} \tag{13}$$

At the root, for symmetrical loading, we take $(c_lc/C_L\bar{c})_{m-1} = (c_lc/C_L\bar{c})_m$ and $e_{m-1} = e_m$; at the tip, we take $(c_lc/C_L\bar{c})_{m+1} = 0$ and $e_{m+1} = 0$. We then obtain the numerical form for the induced incidence from Eqs. (7) and (8) to be

$$\frac{\alpha_i(y)}{C_L\bar{c}} \approx -\frac{1}{4\pi}\sum_{m=1}^{N}\left\{\frac{y^2(y_m+e_m)a_m + y^2b_m + (y_m+e_m)c_m}{y^2-(y_m+e_m)^2} - \frac{y^2(y_m-e_m)a_m + y^2b_r + (y_m-e_m)c_m}{y^2-(y_m-e_m)^2} + \frac{1}{2}ya_m\log\times\left[\frac{(y-e_m)^2-y_m^2}{(y+e_m)^2-y_m^2}\right]^2 + \frac{1}{4}b_m\log\left[\frac{y^2-(y_m+e_m)^2}{y^2-(y_m-e_m)^2}\right]^2 + 2e_ma_m\right\} \tag{14}$$

We then assume the product $c_lc\alpha_i$ also to have a parabolic variation across the strip

$$[(c_lc/C_L\bar{c})(\alpha_i/C_L\bar{c})]_n \approx a_ny^2 + b_ny + c_n \tag{15}$$

The coefficients a_n, b_n, and c_n, are found from Eqs. (9–11), with n replacing m, y_n replacing η_m, and $[(c_lc/C_L\bar{c})\,(\alpha_i/C_L\bar{c})]_n$ replacing $(c_lc/C_L\bar{c})_m$. The numerical form of Eq. (1) is then

§ The assumption of constant values of c_lc and α_i on each strip has resulted in unacceptable errors in the numerical integrations.

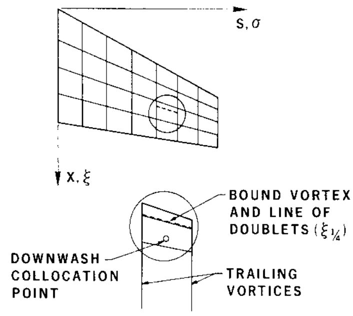

Fig. 1 Surface idealization.

a generalization of Simpson's rule.

$$C_{Diw}/C_L^2 = \left(\frac{4}{A}\right)\sum_{n=1}^{N} e_n\left\{\left[y_n^2 + \left(\frac{1}{3}\right)e_n^2\right]a_n + y_nb_n + c_n\right\} \tag{16}$$

The accuracy of this equation can be obtained by applying it to an elliptic span load. Table 1 presents the error vs the number of spanwise strips.

The spanwise load distribution obtained by the VLM for the rectangular wing leads to a wake integral drag value of $C_{Diw}/C_L^2 = 0.160165$. Since the value $C_{DiV}/C_L^2 = 0.155375$ was obtained directly from the VLM, the correcting scale factor is $C_{Diw}/C_{DiV} = 1.030829$. The scaled spanwise distribution of induced drag is shown in Fig. 2 and is compared to the best results obtained by Garner, Hewitt, and Labrujere.[12] The comparison is excellent.

The second application is the hyperbolic wing also considered by Garner, Hewitt, and Labrujere.[1,12] The wing has a swept planform of constant chord with aspect ratio $A = 4.0$ and has smooth hyperbolic leading and trailing edges with 45° sweepback at the tip, as shown in Fig. 3. Again, a variety of patterns was investigated for the box divisions. Five chordwise divisions were found to be adequate for convergence. Twenty-five spanwise divisions of varying width were found to be adequate, the narrow strips being concentrated near the root and the tip. The idealization of the planform is also shown in Fig. 3. Again the VLM result is low: $C_{DiV}/C_L^2 = 0.076526$. The wake integral gives $C_{Diw}/C_L^2 = 0.083216$, and the correction factor is $C_{Diw}/C_{DiV} = 1.087420$. The corrected spanwise distribution of induced drag is shown in Fig. 3 and is compared with the results of Garner et al.[1,12] (N is the number of chordwise functions in the functional series for the load distribution.) The comparison is seen to be good except near the tip where the results of Refs. 1 and 12 have not achieved convergence. The erratic behavior of the drag distribution shown in Fig. 3, inboard of $y/s = 0.1$, is typical of results obtained for the various idealizations into boxes of this planform. The most smoothly varying distributions were obtained when the strip widths were varied smoothly across the span; large changes in strip width were always accompanied by very erratic behavior of the drag distribution curve, although the spanwise lift distribution was

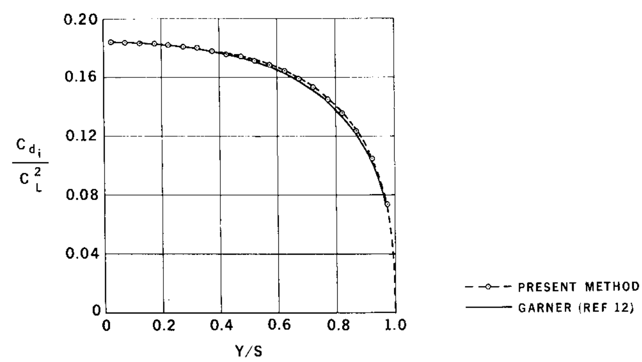

Fig. 2 Spanwise distribution of induced drag on a rectangular wing with $A = 2.0$ at $M = 0$.

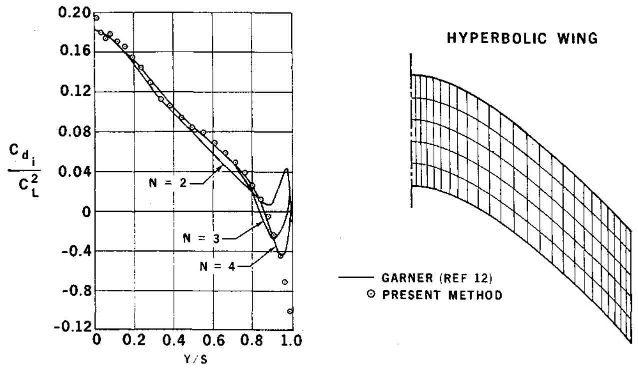

Fig. 3 Spanwise distribution of induced drag on a hyperbolic wing with $A = 4.0$ at $M = 0$.

never affected significantly. The manner of convergence of the drag calculated by the VLM warrants further investigation, but it is apparent that one requirement of the planform idealization is smoothly varying strip widths.

The hyperbolic wing is an example of a planform with smooth leading and trailing edges. Wagner[13] investigated the induced drag distribution on a variable-sweep planform of aspect ratio $A = 4.303$ and with kinked leading and trailing edges. The planform is shown in Fig. 4. A variety of patterns was also investigated for the box divisions for this planform and an idealization that appeared to yield convergence is shown in Fig. 4. The idealization consists of five chordwise boxes and 32 strips of smoothly varying widths concentrated near the root, planform break, and tip. With an angle of attack of one radian, the VLM gives $C_L = 3.024852$ and $C_{DiV} = 0.683045$, or $C_{DiV}/C_L{}^2 = 0.074652$. The wake integral gives $C_{D_{iw}}/C_L{}^2 = 0.076059$ so the spanwise drag correction factor is $C_{D_{iw}}/C_{DiV} = 1.018852$. The corrected spanwise distribution of induced drag is also shown in Fig. 4 and is compared with the results of Wagner.[13] The drag distribution does not exhibit the erratic behavior seen on the hyperbolic wing, but the values near the planform discontinuities are found to be sensitive to the box idealization.

Concluding Remarks

The application of the VLM to the present planar configurations suggests that the method will also be reliable in estimating induced drag distributions on nonplanar configurations such as those whose lifting characteristics are investigated in Ref. 14. The method can also be used for estimating certain quasi-steady rotary derivatives required in Stability and Control Analyses (e.g., the yawing moment coefficient for a wing in steady roll). An unsteady three-dimensional theory of induced drag is necessary for inclusion of rotary effects in flutter analyses of T-tails, because the rocking (rolling) of the horizontal stabilizer occurs at a relatively high reduced frequency. A two-dimensional theory of induced drag (and propulsion) has been presented by Garrick[15] for airfoils oscillating in incompressible flow.

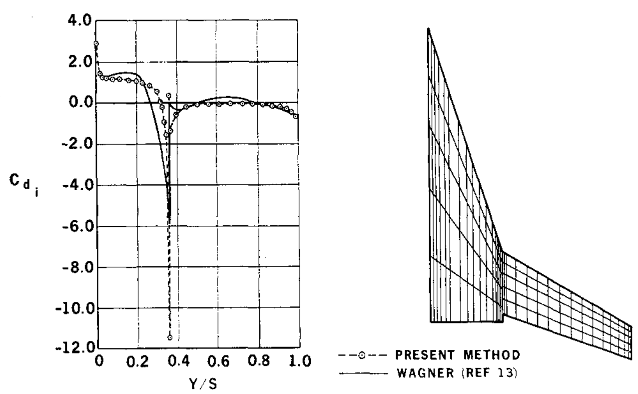

Fig. 4 Spanwise distribution of induced drag on a variable-sweep wing with $A = 4.303$ at $M = 0.23$.

Table 1 Total induced drag error for various numbers of spanwise strips

Number of spanwise strips	Percent error
10	2.56
15	1.02
20	0.28
25	0.01

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